A Multiple Commutator Formula for the Sum of Feynman Diagrams

C.S. Lam[*] and K.F. Liu[†]

[*]*Department of Physics, McGill University,*
3600 University St., Montreal, P.Q., Canada H3A 2T8
Email: Lam@physics.mcgill.ca.
and
[†]*Department of Physics and Astronomy, University of Kentucky*
Lexington, KY 40506, U.S.A.
Email: Liu@ukcc.uky.edu .

Abstract

In the presence of a large parameter, such as mass or energy, leading behavior of individual Feynman diagrams often get cancelled in the sum. This is known to happen in large-N_c QCD in the presence of a baryon, and also in the case of high-energy electron-electron as well as quark-quark scatterings. We present an exact combinatorial formula, involving multiple commutators of the vertices, which can be used to compute such cancellations. It is a non-abelian generalization of the eikonal formula, and will be applied in subsequent publications to study the consistency of large-N_c QCD involving baryons, as well as high-energy quark-quark scattering in ordinary QCD.

1 Introduction

This work arose from the challenge to resolve a possible inconsistency of large-N_c QCD in the presence of baryons [1, 2, 3, 4, 5]. Briefly the problem is as follows. The meson-baryon Yukawa coupling constant is proportional to $\sqrt{N_c}$, so an n-meson baryon tree diagram grows as $N_c^{n/2}$. On the other hand, the complete n-meson baryon amplitude must decrease with N_c like $N_c^{1-n/2}$ [1]. This means that an individual Feynman diagram is a factor N_c^{n-1} larger than the sum of the $n!$ permuted diagrams! To be consistent, a huge cancellation must take place in the sum of diagrams, and the larger n is, the more cancellations are required. Does it really happen? This is the possible inconsistency mentioned above. Similar questions can be asked about baryon-baryon scattering as well.

The examination of this problem led to some very interesting discoveries recently [2, 3, 4]. By looking at the cases of $n = 2$ and $n = 3$, it was found that many new and old quark-model-like relations must be obeyed, and that an infinite tower of baryon resonances with certain characteristics must be present. There are several ways to see the origin of these

relations, one of them is to regard the cancellation requirement as posing constraints that are satisfied by these relations [2]. From that point of view, the larger n is, the more must be cancelled and more constraints have to be satisfied. So there is even a hope of getting more of these interesting relations by understanding the cancellation mechanism for large n.

On the other hand, very little calculable dynamics is put into large-N_c QCD, so these cancellations are quite likely of kinematical origin peculiar to large-N_c QCD, and the predictions may be mostly a result of the Hartree nature of large-N_c baryons [1, 3]. We shall give further support to this point of view in the present and the subsequent paper [6].

The most distinct kinematical feature of a large-N_c baryon is its heavy mass, being of the order of N_c. If a kinematical formula can be found to effect the cancellation in this case, it seems likely that it can also be used in similar kinematical situations, such as the presence of a heavy quark, or fermion-fermion scattering at high energy in which the fermions carry large energy scale. We shall return to these other interesting problems in the future.

We report in this paper a formula for the sum of the $n!$ diagrams, which can be used for these cancellations. We shall refer to it as the *multiple commutator formula* for reasons to be mentioned later. The formula and its predecessors can be formulated as exact combinatorial theorems and for the sake of clarity we will concentrate on these mathematical results in the present paper. Physical applications require approximations at the appropriate kinematical regimes and they will be deferred to future publications [6, 7].

This formula can be considered as a generalization of the 'eikonal formula' [8] useful in the discussion of soft-photon emission [9] and the eikonalization of high-energy scattering processes [10]. The eikonal formula (corollaries 1.1 and 2.1 in section 2) indeed gives a result for the sum of $n!$ diagrams, but it is suitable only for vertices that are effectively numbers. For pion-baryon problem in large-N_c QCD, and high-energy quark-quark scattering in usual QCD, strong-isospin and color are respectively involved, so the vertices are non-abelian and do not commute with one another. To deal with the sum of diagrams in these problems we need a non-abelian generalization of the eikonal formula, which is the multiple commutator formula mentioned above. It is considerably more complicated than the eikonal formula, but in the limit of abelian vertices it reduces to the eikonal formula as it should.

The multiple commutator formula can be derived from a generalization of the eikonal formula, to be called the factorization formula (theorems 1 and 2 of section 2). From that, we derive an intermediate 'folding relation' (theorems 3 and 4), which will be used to derive the multiple commutator formula (theorems 5 and 6). Each of these relations have an on-shell version (theorems 2,3,5) and an off-shell version (theorems 1,4,6). These two versions have identical structures, and in fact the on-shell amplitudes can be obtained in a rather surprising way from the off-shell amplitudes via a 'domino relation' (theorem 7). The statements of these theorems, as well as illustrative examples, will be presented in section 2, and their general proofs will be given in section 3.

The on-shell version of the eikonal formula is in some sense a cancellation formula. It expresses the sum of abelian amplitudes as a product of n energy-δ-functions. In other words, sums of products of propagators are replaced by δ-functions. In high-energy electron-electron scattering via n-photon exchange, individual diagrams go like $s\ln(s)$ for large center-of-mass

energy $\sqrt{s}$, but this leading term is cancelled in the sum of the $n!$ diagrams to yield a final result going only like s [10]. From the point of view of the eikonal formula, this cancellation has to do with the replacement of the propagators by the δ-functions. When it comes to high-energy quark-quark scattering in QCD, the eikonal formula is no longer applicable, and indeed the $s\ln(s)$ contributions are not cancelled, though explicit calculations show that they are concentrated in the channel of color-octet exchange [10]. In this non-abelian situation, the multiple commutator formula replaces the eikonal formula, the sum of the $n!$ non-abelian amplitudes is now given as sums of products of energy-δ-functions and multiple commutators of the vertices. The term that contains pure δ-functions is like the eikonal formula and will lead to the same high-energy cancellations. On the other extreme, there is a term that contains purely multiple commutators without δ-fucntions. This term leads to no reduction in the high energy behavior but the quantum number of the exchange is a pure color octet [7].

In contrast to the case of high energy scattering, the cancellation needed in the large-N_c baryon problem comes from the pure multiple commutator term in the formula. It turns out that each commutator reduces the power of N_c by one [3, 6], so the term with $n-1$ multiple commutators indeed reduces the individual amplitude by a factor N_c^{n-1}, as required for consistency.

To the extent that multiple commutators are involved in the formula, this non-abelian generalization of the eikonal formula bears some superficial resemblance to formulas used in soft-pion productions when current algebra is used [11]. However, the similarity stops there, in that our formula is exact, and is combinatorial and kinematical in nature. No dynamical assumptions are involved, certainly not chiral symmetry nor the presence of a current algebra. The kinematical regime that makes the formula useful is the smallness of the boson energy compared to the mass of the fermion, or the energy it carries, and not compared to some dynamical condensates like f_π as is the case in σ models. Of course our formula can be used in the soft-pion situation as well, but formulas developed there are not general enough for our purposes.

With this brief mention of physical examples to illustrate what the formulas can be used for, we will devote the rest of the paper to purely mathematical results, to explain, illustrate, and prove the stated theorems.

2 Combinatorial Formulas

2.1 Trees and their amplitudes:

It is necessary to establish relatively simple notations to discuss these rather complicated combinatorial problems. To that end, let us denote a tree diagram arranged in the manner of Fig. 1 as $T = [t_1 t_2 \cdots t_n]$. A tree $T = [o]$ with a single boson is just a vertex. If tree $T_1 = [t_{11} \cdots t_{12}]$ is joint to the left of tree $T_2 = [t_{21} \cdots t_{22}]$, the resulting tree $[t_{11} \cdots t_{12} t_{21} \cdots t_{22}]$ will be denoted by $[T_1 T_2]$ or $[T_1 | T_2]$. Note that $[T\emptyset] = [T|\emptyset] = T$ for a null tree $\emptyset$.
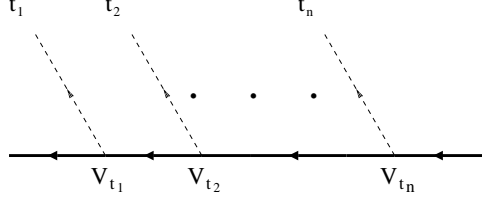
Figure 1:   A standard tree diagram $[t_1 t_2 \cdots t_n]$.

Given a number of trees T_i, $\mathcal{T} = \{T_1; T_2; T_3; \cdots\}$ is defined to be the set of all trees obtained by merging them together in all possible ways. If $N_i = N[T_i]$ is the length (number of boson lines) in tree T_i, then the number of trees in the set $\mathcal{T}$ is given by the multinomial coefficient

$$C(N_1, N_2, N_3, \cdots) = (\sum_i N_i)! / \prod_i N_i! \, , \tag{2.1}$$

and the length of each tree is $\sum_i N_i$. For example, if $T_1 = [12]$ and $T_2 = [345]$, then $\mathcal{T} = \{T_1; T_2\} = \{12; 345\}$ is the set of 10 trees [12345], [13245], [13425], [13452], [31245], [31425], [34125], [31452], [34152], [34512]. In particular, $\{t_1; t_2; \cdots; t_n\}$ is the set of $n!$ trees obtained by permuting the boson lines in the tree $T = [t_1 t_2 \cdots t_n]$. We shall adopt the notation $\{T; \}$ to denote this permutation set of T.

If we join an extra tree T_e to the right of every tree in set $\mathcal{T}$, the resulting set of trees is denoted by $\{T_1; T_2; T_3; \cdots | T_e\}$, or simply $\{\mathcal{T} | T_e\}$. The number of trees in this set is the same as in $\mathcal{T}$, but the length of each tree is $(\sum_i N_i) + N_e$. In the example above, if $T_e = [67]$, then $\{T_1; T_2 | T_e\} = \{12; 345 | 67\}$ consists of the ten trees [1234567], [1324567], $\cdots$, [3415267], [3451267].

Given a tree $T = [t_1 \cdots t_n]$, the 'energy' of boson line t_i in the rest frame of the fermion will be denoted by ω_{t_i}. The 'total energy' of the tree is

$$W[T] = \sum_{i=1}^{n} \omega_{t_i} \tag{2.2}$$

The *off-shell 'amplitude'* of a tree $T = [t_1 t_2 \cdots t_n]$ is *defined* to be the product of n 'propagators',

$$a^*[T] = a^*[t_1 t_2 \cdots t_n] = \prod_{p=1}^{n} \frac{1}{\sum_{j=1}^{p} \omega_{t_j} + i\epsilon} \, , \tag{2.3}$$

and the *on-shell 'amplitude'* of the same tree is obtained by replacing the last 'propagator' by an 'energy-conserving' δ-function:

$$a[T] = a[t_1 t_2 \cdots t_n] = -2\pi i \delta \left(\sum_{j=1}^{n} \omega_{t_j} \right) \prod_{p=1}^{n-1} \frac{1}{\sum_{j=1}^{p} \omega_{t_j} + i\epsilon} \, . \tag{2.4}$$

4

If the tree $T = [o]$ has only one line, then by definition we let $a[o] = -2\pi i\delta(\omega_o)$. Also, $a[To] = -2\pi i\delta\left(W[To]\right) a^*[T]$ for any T.

Incidentally, we put quotes on words like 'amplitude', 'propagator', and 'energy' to avoid confustion when these formulas are used in cases when they are not *exactly* the amplitude, propagator, and energy of the physical process on hand. The theorems proved here are combinatorial and mathematical and the actual physical meaning of quantities like a^*, a and ω_{t_i} are irrelevant. However, it is useful to use these names to give them some physical intuition.

It will also be convenient to adopt the notation $a_E^*[T]$ to mean the off-shell amplitude with 'energy' amount E added to the first boson line t_1 of T:

$$a_E^*[T] = a_E^*[t_1 t_2 \cdots t_n] = \prod_{i=1}^{n} \frac{1}{E + \sum_{j=1}^{i} \omega_{t_j} + i\epsilon} \ . \tag{2.5}$$

The modified on-shell amplitude $a_E[T]$ can be defined similarly. In this notation, it is clear that

$$a^*[T_1 T_2] = a^*[T_1] a_{W[T_1]}^*[T_2] \ , \quad a[T_1 T_2] = a^*[T_1] a_{W[T_1]}[T_2] \ . \tag{2.6}$$

The off-shell amplitude for a set is defined to be the sum of the amplitudes for the individual trees in the set. Thus

$$a^*\{\mathcal{T}\} = \sum_{T \in \mathcal{T}} a^*[T] \ , \tag{2.7}$$

and

$$a^*\{\mathcal{T}|T_e\} = \sum_{T \in \mathcal{T}} a^*[T T_e] \ . \tag{2.8}$$

On-shell amplitudes $a\{\mathcal{T}\}$ and $a\{\mathcal{T}|T_e\}$ for these sets are defined similarly.

2.2　Main Theorems:

We can now state the main combinatorial results, whose proofs are to be found in Section 3. The first result is a *factorization formula* for off-shell amplitudes:
Theorem 1 (off-shell factorization formula):

$$a^*\{T_1; T_2; T_3; \cdots\} = \prod_i a^*[T_i] \ . \tag{2.9}$$

In the special case when every tree $T_i = [t_i]$ is just a single vertex, $\{T; \} = \{t_1; t_2; \cdots; t_n\}$ is the set of $n!$ permuted trees of $T = [t_1 t_2 \cdots t_n]$. Then (2.9) reduces to a known formula which we shall refer to as the 'off-shell abelian eikonal formula':

Corollary 1.1 (off-shell abelian eikonal formula):

$$a^*\{t_1; t_2; \cdots; t_n\} = \prod_{i=1}^{n} a^*[t_i] \ . \tag{2.10}$$

As is well known, this formula can be used for example to show that soft photons emitted from a charged particle satisfies the Poisson distribution.

Example 1: Suppose $T_1 = [1]$ and $T_2 = [23]$. Then (2.9) reads $a^*\{1; 23\} = a^*[1]a^*[23]$. Explicitly, this says

$$\frac{1}{\omega_1(\omega_1 + \omega_2)(\omega_1 + \omega_2 + \omega_3)} + \frac{1}{\omega_2(\omega_2 + \omega_1)(\omega_2 + \omega_1 + \omega_3)} + \frac{1}{\omega_2(\omega_2 + \omega_3)(\omega_2 + \omega_3 + \omega_1)}$$
$$= \frac{1}{\omega_1\omega_2(\omega_2 + \omega_3)} \ , \tag{2.11}$$

an algebraic identitiy that can be verified directly. For simplcity in printing, we have omitted the $+i\epsilon$ which should appear in all the denominators.

There is also a factorization formula for on-shell amplitudes. It states

Theorem 2 (on-shell factorization formula):

$$a\{T_1; T_2; \cdots\} = \prod_j a[T_j] \ . \tag{2.12}$$

Note that the right-hand side vanishes if the energies are generic, *viz.*, when $W[T_i] \neq 0$ for any i, though of course by definition we require $\sum_i W[T_i] = 0$. Thinking of it that way, this theorem tells us what on-shell amplitudes can be added up to cancel one another. Even when the energies are not generic, the presence of additional energy-conserving δ-functions on the right also lead to partial cancellations, at least in high-energy scattering amplitudes. See the remark below Corollary 2.1.

When every tree $T_i = [t_i]$ is just a single vertex, this formula reduces to the 'on-shell abelian eikonal formula': *Corollary 2.1 (on-shell abelian eikonal formula):*

$$a\{t_1; t_2; \cdots; t_n\} = \prod_{j=1}^{n} a[t_j] = \prod_{j=1}^{n} \left(-2\pi i\delta(\omega_j)\right) \ . \tag{2.13}$$

This formula has been used to obtain large cancellations between Feynman diagrams in high-energy electron-electron scattering, resulting thereby an eikonal scattering formula in the sum. The main goal of this paper is to obtain a non-abelian generalization for this formula.

We illustrate this theorem with a simple example.

Example 2: Consider the same trees as in Ex. 1. For these trees, eq. (2.12) demands $a\{1; 23\} = (-2\pi i)^2 \delta(\omega_1 + \omega_2 + \omega_3)\delta(\omega_1)(\omega_2 + i\epsilon)^{-1}$. This identity can be directly verified by using $W_{tot} = \omega_1 + \omega_2 + \omega_3 = 0$:

$$\frac{1}{(\omega_1 + i\epsilon)(\omega_1 + \omega_2 + i\epsilon)} + \frac{1}{(\omega_2 + i\epsilon)(\omega_2 + \omega_1 + i\epsilon)} + \frac{1}{(\omega_2 + i\epsilon)(\omega_2 + \omega_3 + i\epsilon)}$$
$$= \frac{1}{\omega_2 + i\epsilon} \left(\frac{1}{\omega_1 + i\epsilon} + \frac{1}{-\omega_1 + i\epsilon}\right) = -2\pi i\delta(\omega_1)\frac{1}{\omega_2 + i\epsilon} \ . \tag{2.14}$$
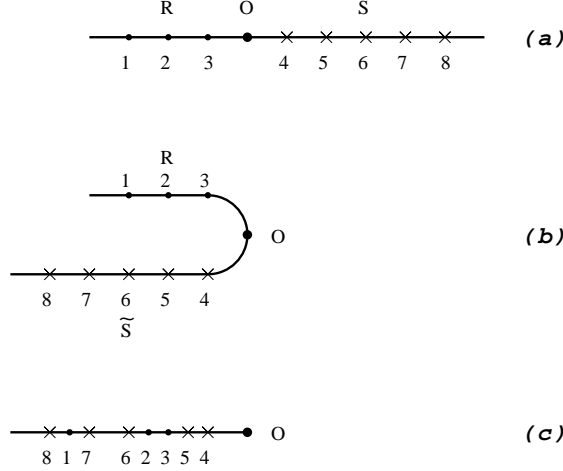
Figure 2: Folding operation for theorems 3 and 4.

We need more notations to go on. The transpose of a tree $T = [t_1 t_2 \cdots t_{n-1} t_n]$ is defined to be the tree $\tilde{T} = [t_n t_{n-1} \cdots t_2 t_1]$. A subtree of $S = [s_1 \cdots s_N]$ from s_p to s_q will be denoted by $\sigma_{p,q} = [s_p s_{p+1} \cdots s_q]$; thus $S = \sigma_{1,N} = [\sigma_{1,k} \sigma_{k+1,N}]$ for any k.

We will consider the tree $[RoS]$ obtained by joining line o to the left of tree S, and then R to the left of the tree $[oS]$. This tree is of length $N[R] + 1 + N[S] \equiv N' + 1 + N$. The next combinatorial formula enables the on-shell amplitude of this tree to be expressed as combinations of on-shell amplitudes in which line o is at the extreme right. This result is needed to obtain the non-abelian generalization of the eikonal formula (2.13).

Theorem 3 (on-shell folding formula):

$$a[RoS] = \sum_{k=0}^{N} (-)^k a\{R; \tilde{\sigma}_{1,k}|o\} a[\sigma_{k+1,N}] . \tag{2.15}$$

For $k = N$, the tree $\sigma_{N+1,N} = \emptyset$ is null, and we should interpret the factor $-2\pi i \delta\left(W[\emptyset]\right)$ simply as 1.

We call this formula the *folding formula* because it can be obtained by a folding operation. Consider first the term $k = N$. This can be obtained by folding the linear tree $[RoS]$ (Fig. 2a) at the point o (Fig. 2b) and merging together the resulting trees R and $\tilde{S}$ in all possible ways to form $a\{R; \tilde{S}|o\}$ (Fig. 2c). For terms with $k < N$, we must tear off the subtree $\sigma_{k+1,N}$ at the end before folding and merging.

Since $a\{R; \tilde{\sigma}_{1,k}|o\} = -2\pi i \delta\left(W[R\tilde{\sigma}_{1,k}o]\right) a^*\{R; \tilde{\sigma}_{1,k}\}$, with the help of (2.9), we may write (2.15) as

Corollary 3.1:

$$a[RoS] = -2\pi i \sum_{k=0}^{N} (-)^k \delta\left(W[R\sigma_{\widetilde{1,k}}o]\right) a[\sigma_{k+1,N}] a^*[R] a^*[\tilde{\sigma}_{1,k}] . \tag{2.16}$$

7

Example 3: Suppose $R = [1]$ and $S = [23]$. Then (2.16) demands

$$a[1o23] = \{-2\pi i\delta(\omega_2 + \omega_3)a[23]a^*[1] + 2\pi i\delta(\omega_3)a^*[1]a^*[2] + a^*[1]a^*[32]\}\Delta , \qquad (2.17)$$

where $\Delta \equiv -2\pi i\delta(\omega_1 + \omega_o + \omega_2 + \omega_3)$. To verify this directly, we use 'energy conservation' $\omega_1 + \omega_o + \omega_2 + \omega_3 = 0$ to rewrite the left-hand side of (2.17) as

$$
\begin{aligned}
a[1o23] &= \frac{1}{(\omega_1 + i\epsilon)(\omega_1 + \omega_o + i\epsilon)(\omega_1 + \omega_o + \omega_2 + i\epsilon)}\Delta \\
&= \frac{1}{(\omega_1 + i\epsilon)(-\omega_2 - \omega_3 + i\epsilon)(-\omega_3 + i\epsilon)}\Delta \\
&= \frac{1}{\omega_1 + i\epsilon}\left(\frac{1}{(\omega_2 + \omega_3 + i\epsilon)} + 2\pi i\delta(\omega_2 + \omega_3)\right)\left(\frac{1}{\omega_3 + i\epsilon} + 2\pi i\delta(\omega_3)\right)\Delta ,(2.18)
\end{aligned}
$$

which gives (2.17).

There is an off-shell extension of the folding theorem:

Theorem 4 (off-shell folding formula):

$$a^*[RoS] = \sum_{k=0}^{N}(-)^k a^*\{R; \tilde{\sigma}_{1,k}|o\}a^*[\sigma_{k+1,N}] . \qquad (2.19)$$

$(a^*[\emptyset] \equiv 1)$. From (2.6) and (2.9), $a^*\{R; \tilde{\sigma}_{1,k}|o\} = a^*[R]a^*[\tilde{\sigma}_{1,k}]\left(W[R\tilde{\sigma}_{1,k}o] + i\epsilon\right)^{-1}$. Hence (2.19) can also be written as

Corollary 4.1:

$$a^*[RoS] = \sum_{k=0}^{N}(-)^k a^*[R]a^*[\tilde{\sigma}_{1,k}]a^*[\sigma_{k+1,N}]/\left(W[R\tilde{\sigma}_{1,k}o] + i\epsilon\right) . \qquad (2.20)$$

Example 4: Using again $R = [1]$ and $S = [23]$, (2.20) demands

$$a^*[1o23] = a^*[1]\left(\frac{a^*[23]}{\omega_1 + \omega_o} - \frac{a^*[2]a^*[3]}{\omega_1 + \omega_2 + \omega_o} + \frac{a^*[32]}{\omega_1 + \omega_3 + \omega_2 + \omega_o}\right) . \qquad (2.21)$$

Explicitly, this requires

$$
\begin{aligned}
&\frac{1}{\omega_1(\omega_1 + \omega_o)(\omega_1 + \omega_o + \omega_2)(\omega_1 + \omega_o + \omega_2 + \omega_3)} \\
&= \frac{1}{\omega_1}\left(\frac{1}{\omega_2(\omega_2 + \omega_3)(\omega_1 + \omega_o)} - \frac{1}{\omega_2\omega_3(\omega_1 + \omega_2 + \omega_o)}\right) \\
&+ \frac{1}{\omega_1}\left(\frac{1}{\omega_3(\omega_3 + \omega_2)(\omega_1 + \omega_3 + \omega_2 + \omega_o)}\right) , \qquad (2.22)
\end{aligned}
$$

which can be directly verified to be true. Again, we have omitted the $+i\epsilon$ which should appear everywhere in the denominators.

We proceed to discuss non-abelian amplitudes. In this case the vertices V_i are matrices that do not commute with one another. We shall define the operator $V[T]$ for a tree $T = [t_1 t_2 \cdots t_n]$ to be the product of its vertices,

$$V[T] = V_{t_1} V_{t_2} \cdots V_{t_n} \ , \tag{2.23}$$

and the multiple commutator of the tree to be

$$V^{MC}[T] = [V_{t_1}[V_{t_2}[\cdots [V_{t_{n-1}}, V_{t_n}] \cdots]]] \ . \tag{2.24}$$

If the tree $T = [t]$ is a single vertex, $V^{MC}[t]$ is taken to mean $V[t] = V_t$.

Recall that $\{T; \} = \{t_1; t_2; \cdots; t_n\}$ is the set of $n!$ trees obtained by permuting the lines of a single tree $T = [t_1 t_2 \cdots t_n]$. The 'non-abelian on-shell amplitude' is defined to be

$$\sum_{K \in \{T;\}} a[K] V[K] \equiv aV\{T; \} \ . \tag{2.25}$$

The off-shell non-abelian amplitude $a^* V\{T; \}$ is defined similarly. Our aim is to find a generalization of (2.13) suitable for the non-abelian amplitudes. Naturally for on-shell amplitudes this generalization must reduce to (2.13) when all $V_{t_i} = 1$.

To state the result we must first establish some convenient ordering of the numbers t_i appearing in the trees of $\{T; \}$. We shall choose the natural order of the original tree T, namely, $t_1 < t_2 < \cdots < t_n$. In terms of this ordering, a partition will be established for every tree K of $\{T; \}$ in the following way. If the rightmost element of K is larger than any other element to its left, then there is one partition and it is the whole tree K. Otherwise, come to the first number larger than it and draw a partition just before that number. Now start again just to the left of the new partition and repeat the same procedure. If this number is now larger than all others to its left, there will be no more partitions. Otherwise another partition will appear just before the first number encountered which is larger than it. Repeat this procedure until the whole tree K is partitioned.

For example, suppose $T = [1234567]$ so the ordering is $1 < 2 < 3 < 4 < 5 < 6 < 7$. The partitions of the following sample trees K are shown to the right of the arrow: $[1234567] \rightarrow [1234567]$, $[1437562] \rightarrow [14|37|56|2]$, $[1357246] \rightarrow [1357|246]$, $[7654321] \rightarrow [7|6|5|4|3|2|1]$.

Let $K = [K_1|K_2|\cdots|K_m]$ be the partition of a tree in $\{T; \}$. Then

Theorem 5 (on-shell multiple commutator formula):

$$\begin{aligned} aV\{T; \} &\equiv \sum_{K \in \{T;\}} a[K] V[K] \\ &= \sum_{K \in \{T;\}} \left(\prod_{a=1}^{m} a[K_a] \right) V^{MC}[K_1] V^{MC}[K_2] \cdots V^{MC}[K_m] \ . \end{aligned} \tag{2.26}$$

In the abelian limit when $V_{t_i} = 1$ for every i, $V^{MC}[K_a] = 0$ unless K_a is a single vertex $[t_a]$, whence $V[K_a] = V[t_a] = 1$. The only tree in $\{T; \}$ that fulfils this condition is the tree $\tilde{T} = [t_n t_{n-1} \cdots t_2 t_1]$, whose partition is $[t_n|t_{n-1}|\cdots|t_2|t_1]$. This is the only tree that

9

contributes to the right of (2.26) and its contribution is $\prod_{i=1}^{n} a[t_i]$. In the present case the left-hand side of (2.26) is $a\{T;\} = a\{t_1; t_2; \cdots; t_n\}$, so eq. (2.26) reduces to (2.13).

For generic energy configurations, where the sum of all energies ω_{t_i} vanishes but the sum of any subset does not, $a[K_i] = 0$ unless K_i is the whole tree K. These trees with one partition are precisely those with t_n at the end, and there are $(n-1)!$ of them. Using the notation T' to denote the tree $[t_1 t_2 \cdots t_{n-1}]$, and $\{T';\}$ to denote its permutation set of $(n-1)!$ trees, one gets

Corollary 5.1: For generic energy configurations,

$$aV\{T;\} = \sum_{J \in \{T';\}} a[Jt_n] V^{MC}[Jt_n] \ . \tag{2.27}$$

This corollary can be used to establish the consistency of scattering in the large-N_c limit. Take for example the case of n mesons scattered from a baryon. It is known that for $N_c \gg 1$, individual tree diagrams grow like $N_c^{n/2}$ but the sum of the $n!$ diagrams must decline like $N_c^{1-n/2}$. This calls for a cancellation of $n-1$ powers of N_c in the sum! It is shown elsewhere [6] that eq. (2.27) can be used to establish this cancellation.

For QCD as well as QED scattering amplitudes in the high-energy limit, dominant contributions often get cancelled between diagrams. Theorem 5 can also be used to compute such cancellations.

Example 5.1: We will use (2.15) to compute directly $aV\{T;\}$ for $T = [12]$. According to that formula, $a[21] = a[1]a[2] - a[12]$. Hence

$$aV\{T;\} = a[12]V_1V_2 + a[21]V_2V_1 = a[12][V_1, V_2] + a[1]a[2]V_2V_1 \ , \tag{2.28}$$

which agrees with (2.26).

Example 5.2: We proceed to compute the more difficult case of $T = [123]$. Then

$$\begin{aligned} aV\{T;\} \ &= a[123]V_1V_2V_3 + a[132]V_1V_3V_2 + a[213]V_2V_1V_3 \\ &\quad + a[231]V_2V_3V_1 + a[312]V_3V_1V_2 + a[321]V_3V_2V_1 \ . \end{aligned} \tag{2.29}$$

Using (2.15), one gets

$$\begin{aligned} a[132] \ &= \ a[13]a[2] - a[1; 2|3] = a[13]a[2] - a[123] - a[213] \ , \\ a[231] \ &= \ a[23]a[1] - a[2; 1|3] = a[23]a[1] - a[213] - a[123] \ , \\ a[312] \ &= \ a[3]a[12] - a[1|3]a[2] + a[21|3] = a[3]a[12] - a[13]a[2] + a[213] \ , \\ a[321] \ &= \ a[3]a[21] - a[2|3]a[1] + a[12|3] = a[3]a[21] - a[23]a[1] + a[123] \\ &= \ a[3] \left(a[1]a[2] - a[12] \right) - a[23]a[1] + a[123] \ . \end{aligned} \tag{2.30}$$

Substituting (2.30) into (2.29) and grouping the terms into multiple commutators, one gets

$$\begin{aligned} &a[123][V_1[V_2, V_3]] + a[213][V_2[V_1, V_3]] + a[3]a[12]V_3[V_1, V_2] \\ + \ &a[13]a[2][V_1, V_3]V_2 + a[23]a[1][V_2, V_3]V_1 + a[3]a[2]a[1]V_3V_2V_1 \ . \end{aligned} \tag{2.31}$$

10

This agrees with (2.26) because the correct partitions of the six trees in $\{T;\}$ are $[123] \rightarrow [123]$, $[213] \rightarrow [213]$, $[312] \rightarrow [3|12]$, $[132] \rightarrow [13|2]$, $[231] \rightarrow [23|1]$, $[321] \rightarrow [3|2|1]$.

There is an off-shell version of Theorem 5:

Theorem 6 (off-shell multiple commutator formula):

$$
\begin{aligned}
a^*V\{T;\} &\equiv \sum_{K\in\{T;\}} a^*[K]V[K] \\
&= \sum_{K\in\{T;\}} \left(\prod_{a=1}^{m} a^*[K_a]\right) V^{MC}[K_1]V^{MC}[K_2]\cdots V^{MC}[K_m] \ .
\end{aligned}
\tag{2.32}
$$

We have discussed three off-shell theorems and three on-shell theorems, and in each case the off-shell relations are identical to the on-shell relations. This is somewhat surprising because left-hand side of these relations is linear in the amplitude, but the right-hand side is not. Starting from an off-shell amplitude, we can replace its last propagator by an energy-conservation δ-function to turn it into an on-shell amplitude. When this is done on the left-hand side, somehow the off-shell amplitudes on the right must all turn into on-shell amplitudes as well, in spite of the non-linearity appearing on the right. This domino effect can be expressed as

Theorem 7 (domino relation):

$$
2\pi\delta\left(\sum_i W[T_i]\right)\lim_{\epsilon\to 0}\left(N\epsilon\prod_i a^*[T_i]\right) = \prod_i a[T_i] \ ,
\tag{2.33}
$$

where $N = \sum_i N[T_i]$ is the total number of boson lines in all the trees.

To demonstrate this relation we must pay special care to the $i\epsilon$'s. As they are defined in (2.3) and (2.4) it is meant to be true when $\epsilon \to 0$. However, expressions like (2.9) and (2.11) are also true for *finite* ϵ's provided an $i\epsilon$ is added to *each* ω_i. This means that we should really have written $ip\epsilon$ instead of $i\epsilon$ in (2.3) and (2.4). In particular, the $i\epsilon$ associated with the last propagator of $a\{T_1; T_2; \cdots; \}$ is actually $iN\epsilon$. With this in mind, the left-hand side of (2.9) is turned into an on-shell amplitude by multiplying it by

$$
-2\pi i\delta\left(\sum_i W[T_i]\right)\left(\sum_i W[T_i] + iN\epsilon\right) = 2\pi N\epsilon\delta\left(\sum_i W[T_i]\right) \ .
\tag{2.34}
$$

When this factor is multiplied to both sides of (2.9) and the result compared with (2.12), Theorem 7 is obtained.

Example 7: We demonstrate explicitly here the special case $T_i = [t_i]$, so $\{T_1; T_2; \cdots\} = \{t_1; t_2; \cdots; t_n\}$. To obtain (2.33), we need to verify that

$$
\lim_{\epsilon\to 0}(in\epsilon)\left(\prod_{j=1}^{n-1}\frac{1}{\omega_j + i\epsilon}\right)\left(\frac{1}{-\sum_{j=1}^{n-1}\omega_j + i\epsilon}\right) = \prod_{j=1}^{n-1}[-2\pi i\delta(\omega_j)] \ .
\tag{2.35}
$$

11

One way of proving it is to take any function $f(\omega_1, \omega_2, \cdots, \omega_{n-1})$, analytic in the lower ω_j planes, and vanishs sufficiently fast at infinity to allow the integration of

$$(-in\epsilon) \int_{-\infty}^{\infty} \prod_{j=1}^{n-1} (\emptyset 1\omega_j + i\epsilon d\omega_j) \, f(\omega_1, \omega_2, \cdots, \omega_{n-1}) \frac{1}{\sum_{j=1}^{n-1} \omega_j - i\epsilon} \qquad (2.36)$$

to be carried out by completing contours in the lower-half planes. The presence of the factor $F = (\sum_{j=1}^{n-1} \omega_j - i\epsilon)^{-1}$ produces a pinch which ensures the integral not to vanish. The result of these $(n-1)$ integrations is to produce a factor $(-2\pi i)^{n-1}$ and to put every $\omega_j = -i\epsilon$, whence $F = (-in\epsilon)^{-1}$, so (2.33) is immediately seen to be true.

3 Proofs

The theorems stated in Section 2 will be proved here.

3.1 Proof of Theorem 1

The easiest way to prove it is to use the Schwinger representation to write the off-shell amplitude for a tree $T = [t_1 t_2 \cdots t_n]$ as

$$\begin{aligned} a^*[T] &= (-i)^n \int_T d^n\tau \exp\left(i \sum_{i=1}^n \omega_{t_i} \tau_{t_i}\right) \\ \int_T d^n\tau &\equiv \int_0^\infty d\tau_{t_n} \int_{\tau_{t_n}}^\infty d\tau_{t_{n-1}} \cdots \int_{\tau_{t_2}}^\infty d\tau_{t_1} \ . \end{aligned} \qquad (3.1)$$

An integration variable τ_{t_i} has been introduced for every boson line, and these variables are ordered ($\infty > \tau_{t_1} \geq \tau_{t_2} \geq \cdots \geq \tau_{t_n} \geq 0$) precisely the same way the boson lines in the tree $T = [t_1 t_2 \cdots t_n]$ are ordered.

Given this relation between the integration region and the ordering of the (boson lines in the) tree, eq. (2.9) is almost obvious. For any $T \in \mathcal{T}$, the ordering of integration variables in $a^*[T]$ is determined by the ordering of $T \in \mathcal{T}$. Thus the variables within each tree T_i are ordered by those individual trees, whatever $T \in \mathcal{T}$ is. The relative orderings of the variables for different T_i's, however, depend on T, but when summed over all $T \in \mathcal{T}$, every possible relative orderings are included, so the union of the integration regions factorizes into the product of the integration regions of $[T_i]$ for different i's. Hence (2.9) is proved.

3.2 Proof of Theorem 2

The proof is almost identical to that of Theorem 1. Instead of (3.1), the on-shell amplitude $a[T]$ is given by

$$\begin{aligned} -2\pi i \delta(W[T]) a[T] &= (-i)^n \int_{T_o} d^n\tau \exp\left(i \sum_{i=1}^n \omega_{t_i} \tau_{t_i}\right) \\ \int_{T_o} d^n\tau &\equiv \int_{-\infty}^\infty d\tau_{t_n} \int_{\tau_{t_n}}^\infty d\tau_{t_{n-1}} \cdots \int_{\tau_{t_2}}^\infty d\tau_{t_1} \ . \end{aligned} \qquad (3.2)$$

12

This differs from (3.1) only in the integration region T_o, which is now defined by the ordering $\infty > \tau_{t_1} \geq \tau_{t_2} \geq \cdots \geq \tau_{t_n} > -\infty$. The rest of the argument goes similarly. When summed over all $T \in \mathcal{T} = \{T_1; T_2; \cdots\}$, the integration variables τ_{t_a} retains only the ordering within each individual tree T_i, and for each tree they integrate from $-\infty$ to $+\infty$. Using (3.2) again for indiviual trees T_i, we obtain (2.12).

3.3   Proof of Theorem 3

The proof can be established by induction on the length $N = N[S]$ of the tree S. If $N = 0$, (2.15) is trivially true. Now assume the formula to be true up to $N = m-1$ for an arbitrary tree R, We must show it to be true also for $N = m$.

Let $S = [s_1 s_2 \cdots s_m]$. It will be useful in the following to divide this into two subtrees $S = [\sigma_{1,k}\sigma_{k+1,m}]$, with $\sigma_{p,q} = [s_p \cdots s_q]$. Let $T_1 = [Ro], T_2 = [S]$, and $\mathcal{T} = \{T_1; T_2\}$. From (2.12),

$$a\{Ro; s_1 s_2 \cdots s_m\} = \sum_{T \in \mathcal{T}} a[T] = a[Ro]a[S] . \tag{3.3}$$

The sum over T can be organized according to the number k of boson lines in S appearing to the left of line o, as follows:

$$\sum_{T \in \mathcal{T}} a[T] = a[RoS] + \sum_{k=1}^{m} a\{R; \sigma_{1,k}|o\sigma_{k+1,m}\} . \tag{3.4}$$

For $k \geq 1$, there are at most $(m-1)$ lines to the right of o, so we can use the induction hypothesis to write

$$a\{R; \sigma_{1,k}|o\sigma_{k+1,m}\} = \sum_{\ell=0}^{m-k} (-)^{\ell} a\{R; \sigma_{1,k}; \tilde{\sigma}_{k+1,k+\ell}|o\}a[\sigma_{k+\ell+1,m}] . \tag{3.5}$$

Substituting this into (3.4) and (3.3), and introducing the variable $k' = k + \ell$, one recovers (2.15) for $N = m$ as follows:

$$\begin{aligned} a[RoS] &= a[Ro]a[S] - \sum_{k'=1}^{m} (-)^{k'} a[\sigma_{k'+1,m}] \sum_{k=1}^{k'} (-)^{k} a\{R; \sigma_{1,k}; \tilde{\sigma}_{k+1,k'}|o\} \\ &= a[Ro]a[S] + \sum_{k'=1}^{m} (-)^{k'} a\{R; \tilde{\sigma}_{1,k'}|o\}a[\sigma_{k'+1,m}] \\ &= \sum_{k'=0}^{m} (-)^{k'} a\{R; \tilde{\sigma}_{1,k'}|o\}a[\sigma_{k'+1,m}] . \end{aligned} \tag{3.6}$$

In reaching this conclusion, we have used the relation

$$\sum_{k=0}^{k'} (-)^{k} a\{R; \sigma_{1,k}; \tilde{\sigma}_{k+1,k'}|o\} = 0 \tag{3.7}$$

13

which can be proven as follows. Define $a < b$ to mean line a lying to the left of line b in a tree. Every tree in $\mathcal{T}_k \equiv \{R; \sigma_{1,k}; \tilde{\sigma}_{k+1,k'}\}$ obeys the order $s_1 < s_2 < \cdots < s_k \circ s_{k+1} > s_{k+2} > \cdots > s_{k'}$, where $s_k \circ s_{k+1}$ means that both $s_k < s_{k+1}$ and $s_k > s_{k+1}$ are allowed. In the former case, $(-)^k a\{\mathcal{T}_k\}$ is cancelled by a similar term in $(-)^{k+1} a\{\mathcal{T}_{k+1}\}$, and in the latter case, it is cancelled by a similar term in $(-)^{k-1} a\{\mathcal{T}_{k-1}\}$, as long as $1 \leq k \leq k' - 1$. Moreover, the terms in $k = 0$ and completely cancelled by terms in $k = 1$, and terms in $k = k'$, are completely cancelled by terms in $k = k' - 1$. We have thus established (3.7), hence (3.6) and the induction proof.

3.4   Proof of Theorem 4

The proof is identical to that of Theorem 3, with a replaced by a^*, and (2.15) replaced by (2.19).

3.5   Proof of Theorem 5

We shall refer to amplitude of the type $\prod_{i=1}^m a[K_i]$ appearing on the right of (2.26) as *derived amplitudes*. Each of its factors $a[K_i]$ will be called an *allowed factor*. Given a subtree S of a tree in $\{T;\}$, its amplitude $a[S]$ is an allowed factor iff the last element of S is larger than all its other elements. A derived amplitude is a product of allowed factors for non-overlapping subtrees.

There is a derived amplitude for each tree $K \in \{T;\}$, so all together there are $n!$ distinct derived amplitudes. The amplitude $a[K]$ of a tree is not equal to its derived amplitude, but by using the folding formula (2.15) it can be expanded as a linear combination of the $n!$ derived amplitudes, with coefficients either $-1, 0$, or $+1$. To carry out the expansion, first locate the largest element o in K and use (2.15) to fold $a[K]$ about o. This gives rise to sums of product of two amplitudes, the first of which is always an allowed factor because of the choice of o being the largest element in K. The second amplitude in the product may not be an allowed factor, but if it is not, we can locate its largest element and use (2.15) to fold it about this element again. Continuing thus, we can reduce $a[K]$ to a sum of products of allowed factors, and hence a sum of derived amplitudes. Examples 5.1 and 5.2 in Section 2 contain explicit examples of how this is done.

We also need to know where every derived amplitude and its allowed factors $a[K_a]$ come from. If they come from a tree L, then every K_a is obtained from a section of L by folding over the largest element in that section. To ask all allowed configurations of that section is to ask all the possible ways to unfold K_a about its end point. This is essentially doing Fig. 2 in reverse, starting from the tree K_a in Fig. 2c with its end point being o, then proceed to separate the rest of the tree K_a into two distinct parts R and $\tilde{S}$ as in Fig. 2b, and finally unfolding it into the tree $RoS \equiv L_a$ as in Fig. 2a, and this L_a is then a possible configuration for a section of L.

We still have to know in what order these different sections L_a are joined together to form the tree $L \in \{T;\}$. If we read back two paragraphs ago the description how $a[L]$ is

folded to get the derived amplitudes, it is clear that K_1 has to be the section of L containing the largest number t_n, and K_2 the section containing the largest number of L outside of K_1, etc. Given that, it is now also clear that the tree $K = [K_1|K_2|\cdots|K_m]$ is a possible tree L. Moreover, these K_a are simply the partitions of K stated just before Theorem 5.

We can put together now all the ingredients for the proof of Theorem 5. Start from the non-abelian amplitude $aV\{T;\} = \sum_{L\in\{T;\}} a[L]V[L]$ and use the unfolding theorem to expand $a[L]$. Next, group the result by the derived amplitudes $\prod_a a[K_a]$, and sum over L. The coefficient for each $a[K_a]$ is just

$$\sum_{L_a\in\mathcal{U}[K_a]} \eta[L_a]V[L_a] = V^{MS}[K_a] \ . \tag{3.8}$$

That this sum is indeed the multiple commutator follows immediately from the various definitions. Now putting all the different a together in the order discussed in the last paragraph, we finally arrive at the right-hand side of (2.26), and hence Theorem 5.

3.6 Proof of Theorem 6

The proof is identical to Theorem 5, except that a must be replaced by a^*, and the off-shell version of the folding theorem (Theorem 4) must now be used.

4 Acknowledgements

References

[1] G. 't Hooft, *Nucl. Phys. B* **160** (1974) 461; E. Witten, *Nucl. Phys. B* **160** (1979) 57; S. Coleman, Erice lecture notes (1979).

[2] J.-L. Gervais and B. Sakita, *Phys. Rev. Lett.* **52** (1984) 87; R. Dashen and A.V. Manohar, *Phys. Lett.* **B315** (1993) 425, 438; E. Jenkins, *Phys. Lett.* **B315** (1993) 431, 441, 447; R.F. Dashen, E. Jenkins and A.V. Manohar, *Phys. Rev. D* **49** (1994) 4713.

[3] C. Carone, H. Georgi, and S. Osofsky, *Phys. Lett.* **B332** (1994)227; M.A. Luty and J. March-Russell, *Nucl. Phys.* **B426** (1994) 71; M.A. Luty, *Phys. Rev. D* **51** (1995) 2322; M.A. Luty, J. March-Russell and M. White, *Phys. Rev. D* **51** (1995) 2332; R.F. Dashen, E. Jenkins and A.V. Manohar, *Phys. Rev. D* **51** (1995) 3697.

[4] M.P. Mattis and M. Mukerjee, *Phys. Rev. Lett.* **61** (1988) 1344; M.P. Mattis and E. Braaten, *Phys. Rev. D* **39** (1989) 2737; P.B. Arnold and M.P. Mattis, *Phys. Rev. Lett.* **65** (1990) 831; M.P. Mattis, *Phys. Rev. D* **51** (1995) 3267.

[5] Y.G. Liang, B.A. Li, K.F. Liu, and R.K. Su, *Phys. Lett.* **B243** (1990) 133.

[6] C.S. Lam, 'Soft interaction of heavy mesons', Proceedings of the Heavy Quark Workshop, Institute of Theoretical Physics, Beijing, August, 1995; C.S. Lam and K.F. Liu, to be published.

[7] Y.J. Feng, O. Hamidi-Ravari, and C.S. Lam, 'Cut diagrams for high energy scatterings', to be published.

[8] H. Cheng and T.T. Wu, *Phys. Rev. Lett.* **22** (1969) 666; *Phys. Rev.* **182** (1969) 1868, 1899; M. Levy and J. Sucher, *Phys. Rev.* **186** (1969) 1656.

[9] D.R. Yennie, S.C. Frautschi, and H. Surra, *Annals of Phys.* **13** (1961) 379; G. Grammer and D. Yennie, *Phys. Rev.* **D8** (1973) 4332.

[10] H. Cheng and T.T. Wu, Expanding Protons: Scattering at High Energies (MIT Press, 1987).

[11] S. Weinberg, *Phys. Rev. Lett.* **16** 879 (1966); *Phys. Rev.* **177** (1969) 2604; *Phys. Rev. D* **2** (1970) 674, 3085; *Phys. Rev. Lett.* **65** (1990) 1177; L. Brown, *Phys. Rev. D* **2** (1970) 3083.